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Continuity of homomorphisms into complete normed algebraic algebras

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Abstract

We prove that, if A and B are complete normed non-associative algebras, and if B is strongly semisimple and algebraic, then dense range homomorphisms from A to B are continuous. As a consequence, we obtain that homomorphisms from complete normed algebras into a complete normed power-associative algebraic algebra B are continuous if and only if B has no nonzero elements with zero square. We also show that homomorphisms from complete normed algebras into a finite-dimensional algebra B are continuous if and only if B has no nonzero elements with zero square.

1. Introduction

A celebrated theorem of C.E. Rickart asserts that, if A and B are complete normed associative algebras, and if B is strongly semisimple, then dense range homomorphisms from A to B are continuous (see, for instance, [10, Theorem 6.18]). It is an open problem if Rickart's theorem remains true whenever the assumption of associativity for the algebras A and B is altogether removed. It is known that associativity of the algebras A and B can be relaxed to power-associativity in Rickart's assertion, without perturbing its validity [8, Theorem 7.3]. In this paper we prove a variant of the result of [8] just quoted. Indeed,

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we show that the assumption of associativity for the algebras A and B can be removed in Rickart's theorem whenever we additionally require that B is algebraic (Theorem 1). Then, through a purely algebraic structure result for power-associative commutative algebras of bounded degree (Proposition 2), we derive that all homomorphisms from complete normed algebras into a complete normed power-associative algebraic algebra B are continuous if and only if B has no nonzero elements with zero square (Theorem 2). In similar terms we characterize those finite-dimensional algebras B having the property that all homomorphisms from complete normed algebras into B are continuous (Theorem 3). We note that Theorem 2 quoted above extends a previous result of the authors [2] in the case that the complete normed algebra B is actually quadratic.

2. A non-associative variant of Rickart's dense-range-homomorphism theorem

Algebras arising throughout this paper are not assumed to be associative. For an element y in an algebra, we define the sequence $\{y^n\}_{n\geqslant 1}$, of left powers of y, by $y^1 = y$ and $y^{n+1} = yy^n$. By $\mathbb{F}$ we mean the field of real or complex numbers.

Lemma 1. *Let A be a complete normed algebra over $\mathbb{F}$, and z an element of A such that $\|z\| < 1$. Then there exists t in A satisfying $zt = t - z$.*

Proof. Consider the element t of A given by $t := \sum_{n=1}^{\infty} z^n$. □

The next result is a particular case of [2, Lemma 1]. For the sake of completeness we include here a proof.

Lemma 2. *Let A be a complete normed algebra over $\mathbb{F}$, B a simple algebra over $\mathbb{F}$ with a unit $\mathbf{1}$, and $\varphi : A \to B$ a surjective homomorphism. Then $\operatorname{Ker}(\varphi)$ is closed in A.*

Proof. Since B is a simple algebra, and the homomorphism $\varphi : A \to B$ is surjective, $\varphi(\overline{\operatorname{Ker}(\varphi)})$ must be equal to either B or zero. Assume that the first possibility happens. Then we have $\mathbf{1} = \varphi(x)$ for some x in $\overline{\operatorname{Ker}(\varphi)}$. Taking y in $\operatorname{Ker}(\varphi)$ with $\|x - y\| < 1$, putting $z := x - y$, and applying Lemma 1, we find some t in A satisfying $zt = t - z$. Since $\mathbf{1} = \varphi(z)$, we arrive in the contradiction $\varphi(t) = \varphi(t) - \mathbf{1}$. It follows $\varphi(\overline{\operatorname{Ker}(\varphi)}) = 0$. □

By a character on an algebra A over $\mathbb{F}$ we mean a nonzero homomorphism from A to $\mathbb{F}$. It follows from Lemma 2 that characters on complete normed algebras over $\mathbb{F}$ are continuous. This fact is of course previously known. Indeed, homomorphisms from complete normed algebras onto (possibly non-unital) complete normed simple algebras are continuous [7, Remark 3.4.ii]. In any case, the proof of the automatic continuity of characters on complete normed algebras provided above becomes the easiest one that we know.

Following [1], we say that an algebra B is algebraic if every single-generated subalgebra of B is finite-dimensional.

Lemma 3. *Let B be a normed algebraic algebra over $\mathbb{F}$ with a unit $\mathbf{1}$, and C a dense subalgebra of B. Then $\mathbf{1}$ lies in C. Moreover, C is simple whenever B is.*

Proof. Take y in C with $\|\mathbf{1} - y\| < 1$. Then the sequence $\{\mathbf{1} - (\mathbf{1} - y)^n\}_{n \geqslant 1}$ converges to $\mathbf{1}$ and is contained in the subalgebra of B generated by y. Since such a subalgebra is finite-dimensional (hence closed in B) and is contained in C, we obtain $\mathbf{1} \in C$.

Assume that B is simple. Let M be a nonzero ideal of C. Then $\overline{M}$ is a nonzero ideal of B, and hence $\overline{M} = B$. By the first part of the proof, we have $\mathbf{1} \in M$, and therefore $M = C$. $\square$

Proposition 1. *Let A be a complete normed algebra over $\mathbb{F}$, B a normed algebraic simple algebra over $\mathbb{F}$ with a unit $\mathbf{1}$, and $\varphi : A \to B$ a dense range homomorphism. Then, for every x in A, the inequality $1 \leqslant \|\mathbf{1} - \varphi(x)\| + \|x\|$ holds.*

Proof. Regarding φ as a homomorphism from A onto $\varphi(A)$, and keeping in mind Lemma 3, we can assume that φ is actually surjective. Then, by Lemma 2, $\mathrm{Ker}(\varphi)$ is a closed ideal of A, and therefore $A/\mathrm{Ker}(\varphi)$ is a normed algebra over $\mathbb{F}$ in a natural way. Let $\phi : A/\mathrm{Ker}(\varphi) \to B$ be the unique surjective isomorphism such that $\phi \circ \pi = \varphi$ (where $\pi : A \to A/\mathrm{Ker}(\varphi)$ denotes quotient mapping). Then the function $y \to |y| := \|\phi^{-1}(y)\|$ from B to $\mathbb{R}$ is an algebra norm on B. Let x be in A. Take a finite-dimensional subalgebra C of B containing $\varphi(x)$ and $\mathbf{1}$ (for instance, put $C := \mathbb{F}\mathbf{1} + D$, where D denotes the subalgebra of B generated by $\varphi(x)$), and consider the associative algebra $\mathcal{L}(C)$ of all linear operators on C. For T in $\mathcal{L}(C)$ put $r(T) := \lim \|T^n\|^{1/n} = \lim |T^n|^{1/n}$ (where $\|\cdot\|$ and $|\cdot|$ stand for the operator norms on $\mathcal{L}(C)$ relative to the norms $\|\cdot\|$ and $|\cdot|$, respectively, on C), and for y in C denote by T_y the operator of left multiplication by y on C. Since the function $r(\cdot)$ is subadditive on commutative subalgebras of $\mathcal{L}(C)$, we have

$$\begin{aligned} 1 &= r(T_1) \leqslant r(T_1 - T_{\varphi(x)}) + r(T_{\varphi(x)}) \leqslant \|T_1 - T_{\varphi(x)}\| + |T_{\varphi(x)}| \\ &= \|T_{\mathbf{1}-\varphi(x)}\| + |T_{\varphi(x)}| \leqslant \bigl\|\mathbf{1} - \varphi(x)\bigr\| + \bigl|\varphi(x)\bigr| \\ &= \bigl\|\mathbf{1} - \varphi(x)\bigr\| + \bigl\|\phi^{-1}\bigl(\varphi(x)\bigr)\bigr\| = \bigl\|\mathbf{1} - \varphi(x)\bigr\| + \bigl\|\pi(x)\bigr\| \leqslant \bigl\|1 - \varphi(x)\bigr\| + \|x\|. \qquad \square \end{aligned}$$

A (two-sided) ideal M of an algebra B is said to be modular if there exists some element e in B such that $y - ye \in M$ and $y - ey \in M$ for every y in B.

Lemma 4. *Let B be a complete normed algebra over $\mathbb{F}$. Then modular maximal ideals of B are closed.*

Proof. Let M be a modular maximal ideal of B. Choose e in B such that $y - ye \in M$ and $y - ey \in M$ for every y in B. If x is in M and if $\|x - e\| < 1$, then, by Lemma 1, there exists an element t in A satisfying $(e - x)t = t - (e - x)$, so that $e = x + xt + t - et$ lies in M, which leads to the contradiction $M = B$. It follows $\|x - e\| \geqslant 1$ for every x in M, and hence $\overline{M}$ is a proper ideal of B. From the maximality of M we deduce $\overline{M} = M$. $\square$

Let B be an algebra. The strong radical of B is defined as the intersection of all modular maximal ideals of B. We say that B is strongly semisimple whenever the strong radical of B is equal to zero.

Theorem 1. *Let B be a complete normed strongly semisimple algebraic algebra over $\mathbb{F}$. Then dense range homomorphisms from complete normed algebras over $\mathbb{F}$ into B are continuous.*

Proof. Let A be a complete normed algebra over $\mathbb{F}$, and $\varphi : A \to B$ a dense range homomorphism. We are going to show that φ is continuous.

First we assume that the complete normed algebraic algebra B is actually simple with a unit $\mathbf{1}$. Then, since the homomorphism φ has dense range, the separating subspace $\mathcal{S}(\varphi)$ of φ is an ideal of B, and therefore, by the simplicity of B and the Closed graph theorem, it is enough to prove that $\mathbf{1} \notin \mathcal{S}(\varphi)$. But, if this were not the case, we would have $\{x_n\} \to 0$ and $\{\varphi(x_n)\} \to 1$ for some sequence $\{x_n\}$ in A, and could apply Proposition 1 to obtain

$$1 \leqslant \lim\bigl\{\bigl\|\mathbf{1} - \varphi(x_n)\bigr\| + \|x_n\|\bigr\} = 0,$$

which is a contradiction.

Now we relax the assumption above that B is simple with a unit to the one in the statement that B is strongly semisimple. Let $\{x_n\}$ be a sequence in A with $\{x_n\} \to 0$ and $\{\varphi(x_n)\} \to y \in B$. Let M be a modular maximal ideal of B. Then, by Lemma 4, B/M is a complete normed algebra in a natural way. Moreover, denoting by π the natural quotient homomorphism $B \to B/M$, the composition $\pi \circ \varphi$ becomes a dense range homomorphism from A to B/M. Since B/M is an algebraic simple algebra with a unit, it follows from the first part of the proof that $\pi \circ \varphi$ is continuous, and hence $\{(\pi \circ \varphi)(x_n)\} \to 0$. Since $\{(\pi \circ \varphi)(x_n)\} = \{\pi(\varphi(x_n))\} \to \pi(y)$, we deduce $\pi(y) = 0$, or equivalently $y \in M$. Finally, since M is an arbitrary modular maximal ideal of B, and B is strongly semisimple, we obtain $y = 0$. □

The following result will not be applied in what follows. However, it is an easy consequence of Lemmas 3 and 4, and has its own interest.

Corollary 1. *Let B be a normed algebraic algebra over $\mathbb{F}$, and C a dense subalgebra of B. Then the strong radical of C is contained in the strong radical of B. Therefore C is strongly semisimple whenever B is.*

Proof. First we note that the conclusion in Lemma 1 is true for (non-necessarily complete) normed algebraic algebras (since their single-generated subalgebras are complete), and that the conclusion in Lemma 4 is true for those normed algebras fulfilling the conclusion in Lemma 1. Therefore modular maximal ideals of B are closed in B. Let M be a modular maximal ideal of B, and let $\pi : B \to B/M$ denote the quotient homomorphism. Then B/M is a normed algebra in a natural way. Since B/M has a unit and is simple and algebraic, and $\pi(C)$ is a dense subalgebra of B/M, it follows from Lemma 3 that $\pi(C)$ has a unit and is simple. Now, since $\pi(C)$ is isomorphic to $C/(C \cap M)$, we deduce that $C \cap M$ is a modular maximal ideal of C. Therefore the strong radical of C is contained in M. Since M is an arbitrary modular maximal ideal of B, the result follows. □

3. Homomorphisms into complete normed power-associative algebraic algebras

An algebra B is said to be of bounded degree if there exists a natural number m such that every single-generated subalgebra of B has dimension $\leqslant m$. The smallest such a number m is called the degree of B and is denoted by $\deg(B)$.

Lemma 5. *Let B be an algebra over $\mathbb{F}$ of bounded degree. Then every family of pair-wise orthogonal nonzero idempotents in B is finite, with cardinal $\leqslant \deg(B)$.*

Proof. Let $\{e_1, \dots, e_n\}$ be a finite family of pair-wise orthogonal nonzero idempotents in B. Choose pair-wise different nonzero elements $\lambda_1, \dots, \lambda_n$ in $\mathbb{F}$, and put $x := \sum_{i=1}^n \lambda_i e_i$. Then for j in $\mathbb{N}$ we have $x^j := \sum_{i=1}^n \lambda_i^j e_i$. Therefore $\{x^1, \dots, x^n\}$ is a linearly independent system of elements of B. Since this system is contained in the subalgebra of B generated by x, we deduce $n \leqslant \deg(B)$. $\square$

An algebra B is said to be power-associative if every single-generated subalgebra of B is associative. If e is an idempotent in a power associative algebra B over $\mathbb{F}$, then we are provided with a Peirce decomposition

$$B = B_1(e) \oplus B_{1/2}(e) \oplus B_0(e),$$

where $B_k(e) := \{y \in B\colon ey + ye = 2ky\}$ for $k = 1, \frac{1}{2}, 0$ [9, p. 131]. The following result is implicitly contained in the proof of [9, Lemma 5.3].

Lemma 6. *Let B be a power-associative algebra, and e an idempotent in B such that $B_0(e) = 0$. Then for y in $B_{1/2}(e)$ we have $y^3 = 0$.*

An element y of an algebra is said to be isotropic whenever $y \neq 0 = y^2$.

Lemma 7. *Let B be a power-associative commutative algebra over $\mathbb{F}$ of bounded degree and having no isotropic element. Then B has a unit.*

Proof. We may assume $B \neq 0$. Then B has nonzero idempotents. Indeed, the subalgebra of B generated by any nonzero element is a finite-dimensional associative algebra without isotropic elements, and hence, by Wedderburn's theory, has a unit, which becomes a nonzero idempotent in B. Choose a maximal family $\mathcal{F}$ of pair-wise orthogonal nonzero idempotents in B. By Lemma 5 we have $\mathcal{F} = \{e_1, \dots, e_n\}$ for some $n \in \mathbb{N}$. Put $e := \sum_{i=1}^n e_i$, and note that e_i belongs to $B_1(e)$ for $i = 1, \dots, n$. We recall that, since B is commutative, $B_1(e)$ and $B_0(e)$ are orthogonal subalgebras of B (see again [9, p. 131]). If $B_0(e) \neq 0$, then, as happened for B, $B_0(e)$ has a nonzero idempotent, which is orthogonal to e_i for $i = 1, \dots, n$, leading to a contradiction. Now that we know that $B_0(e) = 0$, the absence in B of isotropic elements and Lemma 6 give that $B_{1/2}(e) = 0$, so $B = B_1(e)$, and so e is a unit for B. $\square$

Proposition 2. *Let B be a nonzero power-associative commutative algebra over $\mathbb{F}$ of bounded degree and having no isotropic element. Then B is a finite direct sum of simple ideals with a unit.*

Proof. By Lemma 5, B has a unit $\mathbf{1}$, and hence, by Lemma 7, there exists a finite family of pair-wise orthogonal nonzero central idempotents in B with maximum cardinal. Let $\{e_1, \ldots, e_n\}$ be such a family. We have $\sum_{i=1}^n e_i = \mathbf{1}$, since otherwise, by putting $e := \sum_{i=1}^n e_i$, $\{e_1, \ldots, e_n, \mathbf{1} - e\}$ would be a family of pair-wise orthogonal nonzero central idempotents in B with cardinal equal to $n + 1$. For $i = 1, \ldots, n$, write $M_i := Be_i$, so that M_i is an ideal of B with a unit, and the equality $B = \bigoplus_{i=1}^n M_i$ holds. Assume that M_j is not simple for some $j = 1, \ldots, n$. Without loss of generality we can take $j = n$. Choose a nonzero proper ideal P of M_n. By Lemma 7, P has a unit (say f), which is a central idempotent in M_n [5, Lemma 3.8]. Since M_n is a direct summand of B, f is also a central idempotent in B. Now $\{e_1, \ldots, e_{n-1}, f, e_n - f\}$ is a family of pair-wise orthogonal nonzero central idempotents in B, with cardinal equal to $n + 1$. This is a contradiction. □

Now we are almost ready to state and prove the main result in this section. Before establishing the theorem, let us recall some well-known facts. First we note that, if B is a normed algebra over $\mathbb{F}$ with isotropic elements, then there exists a discontinuous homomorphism from some complete normed, associative, and commutative algebra over $\mathbb{F}$ into B [2, Remark 1]. We also note that complete normed algebraic algebras over $\mathbb{F}$ are of bounded degree (see [3] for the power-associative case, and [4] for the general case). Finally, we remark that, if B is a power-associative algebraic algebra over $\mathbb{F}$, then the algebra obtained by symmetrizing the product of B is also power-associative and algebraic.

Theorem 2. *Let B be a complete normed power-associative algebraic algebra over $\mathbb{F}$. Then the following assertions are equivalent*:

(1) *Homomorphisms from complete normed algebras over $\mathbb{F}$ into B are continuous.*
(2) *Homomorphisms from complete normed, associative, and commutative algebras over $\mathbb{F}$ into B are continuous.*
(3) *B has no isotropic element.*

Proof. In view of the previous comments it is enough to show that (3) implies (1). Assume that B has no isotropic element. Let A be a complete normed algebra over $\mathbb{F}$, and $\varphi : A \to B$ a homomorphism. We are going to show that φ is continuous. First we note that, since all properties of B are inherited by its closed subalgebras, we can replace B with the closure of the range of φ, and suppose without loss of generality that φ has dense range. On the other hand, regarding φ as a homomorphism between the complete normed algebras obtained from A and B by symmetrizing their products, we can also assume that B is commutative. Then B is strongly semisimple because it is of bounded degree and Proposition 2 applies. Now the continuity of φ follows from Theorem 1. □

Following [9, pp. 49–50], we say that an algebra B over $\mathbb{F}$ is quadratic if it has a unit $\mathbf{1}$, $B \neq \mathbb{F}\mathbf{1}$, and for each y in B there exist elements $\tau(y)$ and $n(y)$ in $\mathbb{F}$ such that

$y^2 - 2\tau(y)y + n(y)\mathbf{1} = 0$. Since quadratic algebras are algebraic and power-associative, Theorem 2 contains the fact proved in [2, Theorem 1] that homomorphisms from complete normed algebras into a complete normed quadratic algebra B are continuous if and only if B has no isotropic element.

4. Homomorphisms into finite-dimensional algebras

In [7], the ultra-weak radical of an arbitrary algebra is introduced, and the following result is proved.

Proposition 3 [7, Theorem 3.3]. *Let B be a complete normed algebra over $\mathbb{F}$ whose ultra-weak radical is zero. Then homomorphisms from complete normed algebras over $\mathbb{F}$ onto B are automatically continuous.*

Let B be an algebra over $\mathbb{F}$. In general, the definition of the ultra-weak radical of B is slightly complicated (see [7, Definition 3.2] for details). However, in the finite-dimensional case, such a definition simplifies as follows. Denote by $\mathcal{L}(B)$ the associative algebra over $\mathbb{F}$ of all linear operators on B, and by $\mathcal{M}(B)$ the subalgebra of $\mathcal{L}(B)$ generated by $\{L_x\colon x \in B\} \cup \{R_y\colon y \in B\}$, where, for x in B, L_x (respectively, R_x) stands for the operator of left (respectively, right) multiplication by x on B. Then, when B is finite-dimensional, the ultra-weak radical of B is nothing but the largest ideal of B whose elements x satisfy that $L_x \in \mathrm{Rad}(\mathcal{M}(B))$ and $R_x \in \mathrm{Rad}(\mathcal{M}(B))$, where $\mathrm{Rad}(\cdot)$ means Jacobson radical.

Lemma 8. *Let B be a finite-dimensional algebra over $\mathbb{F}$ without isotropic elements. Then the ultra-weak radical of B is equal to zero.*

Proof. First note that, since $\mathrm{Rad}(\mathcal{M}(B))$ is a radical of associative finite-dimensional algebra, it is nilpotent (say $[\mathrm{Rad}(\mathcal{M}(B))]^k = 0$ for a suitable natural number k). Now assume that there is some nonzero element x in the ultra-weak radical of B. Consider the sequence $\{x_n\}_{n\geqslant 1}$ in the ultra-weak radical of B defined by $x_1 = x$ and $x_{n+1} = x_n^2$, so that we have $L_{x_n} \in \mathrm{Rad}(\mathcal{M}(B))$ for every n. Then the absence of isotropic elements in B gives

$$\begin{aligned} 0 \neq x_{k+1} &= L_{x_k}(x_k) = L_{x_k}L_{x_{k-1}}(x_{k-1}) = \cdots \\ &= L_{x_k}L_{x_{k-1}}\cdots L_{x_1}(x) \in \left[\mathrm{Rad}\left(\mathcal{M}(B)\right)\right]^k(x) = 0, \end{aligned}$$

a contradiction. □

Theorem 3. *Let B be a finite-dimensional algebra over $\mathbb{F}$. Then the following assertions are equivalent*:

(1) *Homomorphisms from complete normed algebras over $\mathbb{F}$ into B are continuous.*
(2) *Homomorphisms from complete normed, associative, and commutative algebras over $\mathbb{F}$ into B are continuous.*
(3) *B has no isotropic element.*

Proof. By the comments before Theorem 2, it is enough to show that (3) implies (1). Suppose that B has no isotropic element. Let A be a complete normed algebra over $\mathbb{F}$, and $\varphi : A \to B$ a homomorphism. Since both the finite dimensionality and the absence of isotropic elements are inherited by all subalgebras of B, there is no loss of generality in assuming that φ is surjective. Then, since B has zero ultra-weak radical (by Lemma 8), the continuity of φ follows from Proposition 3. □

We commented above that the definition in [7] of the ultra-weak radical of an algebra B drastically simplifies in the case when B is finite-dimensional. Now, as a methodological remark, we note that the proof of Proposition 3 given in [7] also has a relevant simplification when the algebra B is finite-dimensional (just the case applied in the proof of Theorem 3). Indeed, since finite-dimensional algebras have uniqueness of (the topology of the) norm, the uniqueness-of-norm theorem for complete normed algebras with zero "weak radical" [7, Theorem 1.10] applied in the proof of the general case of Proposition 3 becomes unnecessary in our setting. By the way, whereas in general we only know the inclusion of the weak radical into the ultra-weak radical, in the finite-dimensional case weak and ultra-weak radicals coincide.

In the case $\mathbb{F} = \mathbb{C}$, a new proof of Theorem 3 can be given. We include here such a new proof because of its own interest.

Lemma 9. *Let B be an algebraic complex algebra without isotropic elements, and M an ideal of B. Then B/M has no isotropic element.*

Proof. Assume that there is some isotropic element $t + M$ in B/M. Denote by C the subalgebra of B generated by t, and choose a basis $\{u_1, \dots, u_n\}$ of $C \cap M$. Then for $i, j, k = 1, \dots, n$ there are complex numbers λ_i, μ_{ij}, and ρ_{ijk} satisfying $t^2 = \sum_{m=1}^n \lambda_m u_m$, $tu_i + u_i t = \sum_{m=i}^n \mu_{im} u_m$, and $u_i u_j = \sum_{m=1}^n \rho_{ijm} u_m$, respectively. Take a nontrivial solution $(\alpha_0, \dots, \alpha_n) \in \mathbb{C}^{n+1}$ of the homogeneous system

$$\alpha_0^2 \lambda_m + \alpha_0 \sum_{i=1}^n \alpha_i \mu_{im} + \sum_{i,j=1}^n \alpha_i \alpha_j \rho_{ijm} = 0 \quad (m = 1, \dots, n)$$

(see, for instance, [6, p. 372]), and put $u := \alpha_0 t + \sum_{m=1}^n \alpha_m u_m$. Then u becomes an isotropic element of B, which is impossible by the assumptions on B. □

New proof of Theorem 3 in the case $\mathbb{F} = \mathbb{C}$. Let B be a finite-dimensional complex algebra without isotropic elements. We show that homomorphisms from complete normed complex algebras into B are continuous, by induction on the dimension of B (say m). If $m = 1$, then $B = \mathbb{C}$, and the desired result is nothing but the well-known automatic continuity of characters. Assume that $m > 1$ and that, for $r < m$, homomorphisms from complete normed complex algebras into r-dimensional complex algebras without isotropic elements are continuous. Let A be a complete normed complex algebra, and $\varphi : A \to B$ a homomorphism. If φ is not surjective, then φ is continuous because of the induction hypothesis. Assume that φ is surjective. If B is simple, then, by [7, Remark 3.4.ii],

φ is continuous. Therefore we can assume in addition that B is not simple, and hence has a nonzero proper ideal (say M). Denote by $\pi : B \to B/M$ the natural quotient homomorphism, and put $\psi := \pi \circ \varphi$. Then, by Lemma 9 and the induction hypothesis, ψ is continuous. Therefore $\mathrm{Ker}(\psi) = \varphi^{-1}(M)$ is a closed ideal of A, and, again by the induction hypothesis, the mapping $\phi : x \to \varphi(x)$ from $\mathrm{Ker}(\psi)$ to M is continuous. Now let $\{x_n\}$ be a sequence in A converging to zero, and such that $\{\varphi(x_n)\}$ converges to some y in B. From the continuity of ψ we deduce that y lies in M. Then, choosing $z \in \varphi^{-1}(y) \subseteq \mathrm{Ker}(\psi)$, we have $\mathrm{Ker}(\psi) \supseteq \{zx_n\} \to 0$, so that the continuity of ϕ gives

$$y^2 = \lim\bigl\{y\varphi(x_n)\bigr\} = \lim\bigl\{\varphi(z)\varphi(x_n)\bigr\} = \lim\bigl\{\phi(zx_n)\bigr\} = 0.$$

Therefore $y = 0$ because B has no isotropic element, and φ is continuous by the Closed graph theorem. □

After Theorems 2 and 3, the following conjecture naturally arises.

Conjecture 1. *Homomorphisms from complete normed algebras over $\mathbb{F}$ into a complete normed algebraic algebra B over $\mathbb{F}$ are continuous if (and only if) B has no isotropic element.*

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